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MAPPINGS OF FINITE DISTORTION: REMOVABLE SINGULARITIES

BY

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ABSTRACT

We show that certain small sets are removable for bounded mappings of finite distortion for which the distortion function satisfies a suitable subexponential integrability condition. We also give an example demonstrating the sharpness of this condition.

1. Introduction

We call a mapping $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ **a mapping of finite distortion** if it satisfies

$$|Df(x)|^n \leq K(x)J(x,f) \quad \text{a.e.},$$

where $K(x) < \infty$, and if also $J(\cdot, f) \in L^1_{\mathrm{loc}}(\Omega)$. Here and in the sequel $\Omega \subset \mathbb{R}^n$ is open, connected, and bounded. The basics of the theory of mappings of finite distortion have been established in the papers [1], [2], [3], [4], [5], [10], [11] and [12]; also see the monograph [7]. In these works it has been demonstrated that (sub)exponential integrability in the sense described below is both sufficient and essentially necessary for the validity of many basic properties similar to those of mappings of bounded distortion, that is, mappings of finite distortion with

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$K \in L^\infty$. However, there are still many other properties of mappings of bounded distortion, also called quasiregular mappings, for which no analog is known in our more general setting; see the monographs [13], [14], [8], [16].

The purpose of this note is to study the question of removable singularities for bounded mappings of finite distortion. Our principal message is that sufficiently small sets are indeed removable under the (sub)exponential integrability assumption on K whereas even a single point can fail to be removable under weaker integrability assumptions.

Let us next describe what we mean by (sub)exponential integrability. Let $\Phi\colon [0,\infty) \to [0,\infty)$ be a strictly increasing, differentiable function. We call such functions Orlicz functions and we make the following two assumptions:

(Φ-1) $\int_1^\infty \frac{\Phi'(t)}{t}\,dt = \infty$,

(Φ-2) $\lim_{t\to\infty} t\Phi'(t) = \infty$.

We will prove our removability results under the assumption that $\exp(\Phi(K))$ is integrable with Φ satisfying the above two conditions. Both of them are needed but the second could be replaced with some other regularity requirement on Φ. Notice that (Φ-1) and the integrability of $\exp(\Phi(K))$ do not even guarantee the L^1-integrability of K. The role of (Φ-2) is to take care of such pathologies; see [12]. It is often automatically guaranteed, as for $\Phi(t) = \lambda t$, $\Phi(t) = t(\log(e+t))^{-1}$, and for most of the functions that are close to being linear (or that grow faster). Our removability theorem will be given in terms of a capacity associated to Φ. In order to introduce this capacity, we first define

$$\psi(t) = t\exp(\Phi(t)). \tag{1.1}$$

Because ψ is strictly increasing, we may define an increasing function $h\colon [0,\infty) \to [0,\infty)$ by setting

$$h(t) = t^n(\psi^{-1}(t^{2n}))^{n-1}. \tag{1.2}$$

We say that a compact subset $E \subset \Omega$ has zero h-capacity, $\mathrm{cap}_h(E) = 0$, if

$$\inf\left\{ \int_\Omega h(|\nabla u|) : u \in C_0^\infty(\Omega), u(x) = 1 \forall x \in G \text{ for some open } G \supset E \right\} = 0.$$

Using the h-capacity, we give the following result.

THEOREM 1.1: *Let Φ and h be as above, such that the assumptions (Φ-1) and (Φ-2) hold. Let $E \subset \Omega$ be a compact set whose h-capacity is zero. If $f\colon \Omega \setminus E \to \mathbb{R}^n$ is a bounded mapping of finite distortion such that*

$$\int_{\Omega\setminus E} \exp(\Phi(K(x))) < \infty,$$

then f extends to a mapping of finite distortion in Ω.

In Section 2 we show that — under the assumptions (Φ-1) and (Φ-2) — each singleton has zero h-capacity. It is then easy to further construct Cantor sets whose h-capacity is zero.

There are previous results related to Theorem 1.1. When $K \in L^\infty$, our claim is the counterpart of the basic result that sets of zero conformal capacity are removable for bounded quasiregular mappings. In that setting also much larger sets are removable; see [6], [8]. In our setting, improvements of that type on Theorem 1.1 appear to require tools that are not yet available; see, however, [1] for the planar case which is somewhat easier. When $\Phi(t) = \lambda t$, the claim of Theorem 1.1 has been proven in [2], [3]. Also see [15] for the removability of a point for homeomorphic mappings when $\Phi(t) = \lambda t$.

As practical examples, assumptions (Φ-1) and (Φ-2) are satisfied for

$$\Phi(t) = t, \ \frac{t}{\log(e+t)}, \ \frac{t}{\log(1+t)\log\log(e^e+t)}, \dots$$

for any string of iterated logarithms. The corresponding capacity functions can (up to a multiplicative constant) be estimated from above as follows:

$$\text{if } \Phi(t) = t \quad \text{then } h(t) \le t^n(\log(e+t))^{n-1},$$

$$\text{if } \Phi(t) = \frac{t}{\log(e+t)} \quad \text{then } h(t) \le t^n(\log(e+t)\log\log(e^e+t))^{n-1}, \dots.$$

One could also formulate Theorem 1.1 in terms of Hausdorff measures arising from general gauge functions associated to h, but we have chosen, for technical reasons, to restrict ourselves to the capacity setting.

Our second result demonstrates the necessity of (Φ-1) for Theorem 1.1.

THEOREM 1.2: *Let Φ be an Orlicz-function such that*

$$\int_1^\infty \frac{\Phi'(s)}{s} ds < \infty. \tag{1.3}$$

Let Q be a closed cube in $\mathbb{R}^n$, centered at the origin. Then there exists a bounded, continuous mapping $f\colon Q \setminus \{0\} \to \mathbb{R}^n$ of finite distortion such that

$$\int_{Q\setminus\{0\}} \exp(\Phi(K(x))) < \infty,$$

but so that f does not extend to a mapping of finite distortion in Q.

Notice that, for example, the following functions Φ do not satisfy assumption (Φ-1) when $\epsilon > 0$ and thus satisfy (1.3) (but do satisfy (Φ-2)):

$$\Phi(t) = \frac{t}{t^\epsilon}, \ \frac{t}{\log^{1+\epsilon}(e+t)}, \ \frac{t}{\log(e+t)\log^{1+\epsilon}\log(e^e+t)}, \dots.$$

We will prove Theorem 1.1 by extending the corresponding argument given in [2], [3] for the case $\Phi(t) = \lambda t$ to our more general setting in Section 2. This partially relies on recent results in [12] but requires some improvements on the prior arguments. The construction for Theorem 1.2 is based on a modification of the constructions given in [10], [12]. This will be explained in Section 3.

2. Proof of Theorem 1.1

Let us define two auxiliary Orlicz-functions:

$$\begin{aligned} \psi(t) &= t\exp(\Phi(t)), \\ g(s) &= \frac{s}{\psi^{-1}(s)} - 1, \ s > 0, \text{ and } g(0) = 0. \end{aligned} \tag{2.1}$$

We notice that ψ is strictly increasing so that the inverse function ψ^{-1} makes sense. We immediately have

$$g(\psi(t)) = \exp(\Phi(t)) - 1. \tag{2.2}$$

We then have the following ([12], lemma 2.1).

LEMMA 2.1: *Assume that Φ satisfies (Φ-1). Then*

(a) $\int_1^\infty \frac{g(s)}{s^2} ds = \infty$ *and*

(b) *given $a, b \geq 0$ we have*

$$g(ab) \leq a + \exp(\Phi(b)) - 1.$$

Recall that the function h was defined as $h(t) = t^n(\psi^{-1}(t^{2n}))^{n-1}$. The following lemma shows that Theorem 1.1 is not empty: singletons have h-capacity zero.

PROPOSITION 2.2: *Let h, Φ and Ω be as above, such that assumptions (Φ-1) and (Φ-2) hold. Then*

$$\operatorname{cap}_h(\{x\}) = 0 \quad \textit{for every } x \in \Omega.$$

We record the following result from [9] that we employ for the proof of Proposition 2.2.

LEMMA 2.3: *If φ: $[0, \infty) \to [0, \infty)$ is decreasing and*

$$\int_1^\infty \varphi^{1/n}(t) dt = \infty,$$

then there exists a radial function $u \in W_0^{1,n}(B(0,1))$ *such that* $u > 0$, u *is continuous in* $B(0,1) \setminus \{0\}$, $u(x) \to \infty$ *as* $|x| \to 0$ *and*

$$\int_{B(0,1)} |\nabla u| \varphi^{(1-n)/n}(|\nabla u|) < \infty.$$

We are now ready to prove Proposition 2.2.

Proof: Without loss of generality, assume that $x = 0$ and $B(0,1) \subset \Omega$. Let $\varphi(t) = (t\psi^{-1}(t^{2n}))^{-n}$. Then φ is decreasing and differentiable in $(0,\infty)$. By change of variables,

$$\begin{aligned}\int_1^\infty \varphi^{1/n}(t)dt &= \int_1^\infty \frac{dt}{t\psi^{-1}(t^{2n})} = \int_1^\infty \frac{1}{t}\Big(\frac{g(t^{2n})}{t^{2n}} + \frac{1}{t^{2n}}\Big)dt \\ &> \int_1^\infty \frac{g(t^{2n})}{t^{2n+1}}dt = \int_1^\infty \frac{g(s)}{2ns^2}ds = \infty.\end{aligned}$$

Thus, by Lemma 2.3, there exists a radial function $u \in W^{1,n}(B(0,1))$, continuous in the punctured unit ball, for which $u(x) \to \infty$ as $|x| \to 0$ and

$$\int_{B(0,1)} h(|\nabla u|) = \int_{B(0,1)} |\nabla u| \varphi^{(1-n)/n}(|\nabla u|) = M < \infty.$$

We can further assume that the support of u is contained in the unit ball. Define $u_k = \min\{\frac{1}{k}u, 1\}$. Since $u \in W_0^{1,n}(B(0,1))$ and grows to infinity as $|x|$ tends to zero, functions u_k are valid test functions for h-capacity for every positive k. For this we also need to know that one can approximate by smooth functions with respect to the Orlicz function h. This density property is known to hold when the function h is doubling, i.e., when $h(2t) \le Ch(t)$ for all $t > 0$, see [8]. Now by assumption (Φ-2), h is doubling for big t, and by the definition of h doubling then holds for all t.

Using the fact that that ψ^{-1} is increasing, we have

$$\begin{aligned}\int_\Omega h(|\nabla u_k|) &= \int_{B(0,1)} h(|\nabla u_k|) = \int_{B(0,1)} |\nabla u_k|^n (\psi^{-1}(|\nabla u_k|^{2n}))^{n-1} \\ &= k^{-n} \int_{B(0,1)} |\nabla u|^n (\psi^{-1}(k^{-2n}|\nabla u|^{2n}))^{n-1} \\ &\le k^{-n} \int_{B(0,1)} |\nabla u|^n (\psi^{-1}(|\nabla u|^{2n}))^{n-1} = k^{-n} \int_{B(0,1)} h(|\nabla u|) \\ &= k^{-n}M \to 0\end{aligned}$$

as $k \to \infty$. ∎

We need two more lemmas. The first one is from [12], Proposition 2.6.

LEMMA 2.4: *Let Φ and g be as above, such that assumptions (Φ-1) and (Φ-2) hold. If $f \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and*

$$\int_\Omega g(|Df|^n) < \infty,$$

then the pointwise Jacobian $J(x,f)$ is locally integrable, and $J(x,f)$ coincides with the distributional Jacobian.

The latter conclusion of the lemma says that we can integrate by parts against the Jacobian, that is,

$$\int_\Omega \phi J(x,f)dx = -\int_\Omega f_i J(x, f_1, \ldots, f_{i-1}, \phi, f_{i+1}, \ldots, f_n)dx$$

for each $i = 1, \ldots, n$ and all $\phi \in C_0^\infty(\Omega)$.

LEMMA 2.5: *Let h, g and Φ be as above, such that assumptions (Φ-1) and (Φ-2) are valid. Then the following holds.*

1. *There exists a constant $C = C(n, \Phi) > 0$ such that for every $a, b \geq 0$*

$$a^{n-1}b \leq g(a^n) + h(b) + C.$$

2. *For every $a \geq 0$ there exists a constant $M = M(a, n, \Phi) > 0$ such that*

$$h(a+t) \leq M + Mh(t) \quad \forall t \geq 0.$$

Proof: 1. We have three cases:

(i) $a \leq b$: Then $a^{n-1}b \leq b^n \leq h(b)$, when b is sufficiently large. Thus we may choose C so that the claim holds in this case.

(ii) $b < a$ and $b < a/\psi^{-1}(a^n)$: Then $a^{n-1}b \leq a^n/\psi^{-1}(a^n) = g(a^n) + 1$.

(iii) $a/\psi^{-1}(a^n) < b < a$: *Claim*: For every $\epsilon > 0$ there exists a constant $L = L(n, \Phi, \epsilon) > 0$ such that $a^\epsilon > \psi^{-1}(a^n)$ for every $a \geq L$.

Proof of the claim: We show that $a^\epsilon/\psi^{-1}(a^n) \to \infty$ as $a \to \infty$. Recall that, by (Φ-2), $t\Phi'(t) \to \infty$ when t tends to infinity. This implies that, for each $s > 0$, there is $t_s > 0$ so that $\Phi(t) \geq s \log t$ for $t > t_s$. Because $\psi(t) = t\exp(\Phi(t))$, we conclude that ψ exceeds each polynomial growth rate for sufficiently large t. The claim follows.

Now, for a large enough, $a^{1/2} < a/\psi^{-1}(a^n) < b$. Because ψ^{-1} is increasing, we have

$$\begin{aligned} a^{n-1}b &\leq b^n(\psi^{-1}(a^n))^{n-1} + C(n,\Phi) \\ &\leq b^n(\psi^{-1}(b^{2n}))^{n-1} + C(n,\Phi) = h(b) + C(n,\Phi). \end{aligned}$$

2. A simple but tedious calculation using the definitions (1.1), (1.2), (2.1) of ψ, g, and h gives us the estimate

$$h'(t) \leq (2(n-1)+n)t^{n-1}(\psi^{-1}(t^{2n}))^{n-1}$$

for all $t > 0$, and using the previous claim we obtain the estimate

$$h'(t) \leq t^n + C$$

for all t where $C = C(n, \Phi)$. Now

$$\begin{aligned} h(a+t) - h(t) = \int_t^{a+t} h'(t) &\leq Ca + a(t+a)^n \\ &\leq Ca + 2^n a^{n+1} + 2^n a t^n \leq M + Mt^n. \end{aligned}$$ ∎

Proof of Theorem 1.1: The idea of the proof is similar to that of the proof of a weaker result given in [3]. By Lemma 2.4 it suffices to prove that

$$\int_F g(|Df|^n) < \infty \tag{2.3}$$

for every compact set $F \subset \Omega$. Indeed, it then follows that $f \in W^{1,1}_{\text{loc}}(\Omega; \mathbb{R}^n)$, and other claims follow by applying Lemma 2.4. To see that (2.3) implies that the coordinate functions f^j are locally integrable, we use the following argument.

Consider the functions $f_i^j = \min\{i, \max\{f^j, -i\}\}$. Then by the definition of g, $|\nabla f^j| \in L^1_{\text{loc}}(\Omega)$ and thus $f_i^j \in W^{1,1}_{\text{loc}}(\Omega)$. For each $x \in \Omega$, consider a ball B_x and a constant M for which $\int_{B_x} |\nabla f_i^j| \leq M$ for all $i \in \mathbb{N}$. Then the Poincaré inequality implies that the sequence $(f_i^j)_i$ is bounded in $W^{1,1}(B_x)$. Recall that one can use the average over any subset A of B_x with $|A| \geq |B_x|/2$ in the Poincaré inequality, and that $|E| = 0$. Monotone convergence now implies that the coordinate functions of f are locally integrable in B_x, and hence over any compact $F \subset \Omega$. Note that although inequality (2.3) is assumed for the entire Ω in Lemma 2.4, it suffices to consider integrals over compact sets, as we only need local conclusions of Lemma 2.4.

Fix a compact set $F \subset \Omega$. Then there exists a test function $\eta \in C_0^\infty(\Omega)$ such that $0 \leq \eta \leq 1$ and $\eta = 1$ in F. Since $\text{cap}_h(E) = 0$, there exists a sequence (ϕ_j) with properties

(i) $\phi_j \in C_0^\infty(\Omega)$ for every $j \in \mathbb{N}$,
(ii) $0 \leq \phi_j \leq 1$ for every $j \in \mathbb{N}$,
(iii) for every $j \in \mathbb{N}$ there exists an open $U_j \supset E$ such that $\phi_j = 1$ in U_j,
(iv) $\lim_{j\to\infty} \phi_j(x) = 0$ for almost every $x \in \Omega$, and

(v) $\lim_{j\to\infty}\int_\Omega h(|\nabla\phi_j|)=0$.

Define $\varphi_j=(1-\phi_j)\eta\in C_0^\infty(\Omega\setminus E)$. We want to show that

$$\int_\Omega g(|\varphi_j Df|^n)\le C<\infty. \tag{2.4}$$

This would prove the theorem, since by the choice of η and Fatou's lemma,

$$\begin{aligned}\int_F g(|Df|^n)\le\int_\Omega g(|\eta Df|^n)&=\int_\Omega\lim_{j\to\infty}g(|\varphi_j Df|^n)\\ &\le\liminf_{j\to\infty}\int_\Omega g(|\varphi_j Df|^n)\le C.\end{aligned}$$

First of all, the function g is, as an Orlicz-function, increasing. Thus we can use the finite distortion property of f to obtain the estimate

$$\int_\Omega g(|\varphi_j Df|^n)\le\int_\Omega g(|\varphi_j|^n J(x,f)K(x)). \tag{2.5}$$

By Lemma 2.1 (b),

$$\int_\Omega g(|\varphi_j|^n J(x,f)K(x))\le\int_\Omega|\varphi_j|^n J(x,f)+\int_\Omega\exp(\Phi(K(x))).$$

By our assumptions, the second term on the right hand side is bounded and the sum of the two terms is finite. Furthermore, Lemma 2.4 allows us to integrate by parts to handle the first term on the right hand side:

$$\int_\Omega|\varphi_j|^n J(x,f)\le n||f||_\infty\int_\Omega|\varphi_j|^{n-1}|\nabla\varphi_j||Df|^{n-1}.$$

The first part of Lemma 2.5 shows that the right hand side is no more than

$$n||f||_\infty\Big(\int_\Omega g(|\varphi_j|^n|Df|^n)+\int_\Omega h(|\nabla\varphi_j|)+C\Big). \tag{2.6}$$

Since we assumed f to be bounded, we may (by scaling f) assume that $n||f||_\infty\le 1/2$. Then the term containing the integral of g in (2.6) can be moved to the left hand side of (2.5) and thus can be forgotten.

Furthermore

$$|\nabla\varphi_j|=|\nabla((1-\phi)\eta)|=|(1-\phi)\nabla\eta-\eta\nabla\phi_j|.$$

Because the function h is increasing we conclude that

$$\int_\Omega h(|\nabla\varphi_j|)\le\int_\Omega h(|(1-\phi)\nabla\eta|+|\eta\nabla\phi_j|)\le\int_\Omega h(|\nabla\eta|+|\nabla\phi_j|). \tag{2.7}$$

Since $|\nabla\eta|$ is bounded, we can apply the second part of Lemma 2.5 to the term on the right hand side of inequality (2.7):

$$\int_\Omega h(|\nabla\eta| + |\nabla\phi_j|) \leq M|\Omega| + M\int_\Omega h(|\nabla\phi_j|).$$

By the choice of the functions ϕ_j, the second term on the right hand side tends to zero. Combining this with preceding discussion results in the inequality (2.4) and the proof is thus complete. ∎

3. Proof of Theorem 1.2

We will modify the construction made in [12] (see also [10]) in order to give the desired mapping. The initial idea for constructions of this type goes back, at least, to [6]. In practise, what we do is to use several suitably modified versions of the mapping given in [12]. We have not found out a slick way to reduce our construction to the existing ones and thus we, for the convenience of the reader, give a rather detailed reasoning.

We will use the cubic norm $\| x\| = \max_i |x_i|$ as our standard norm from now on. Using the cubic norm, the x_0-centered closed cube with edge length $2r > 0$ and sides parallel to coordinate axes can be represented in the form

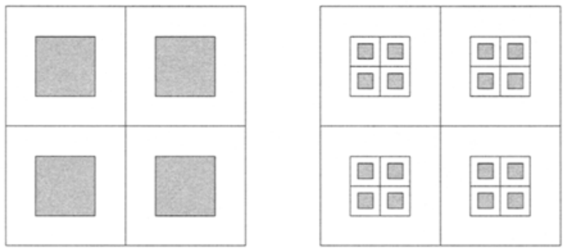
$$Q(x_0, r) = \{x \in \mathbb{R}^n : \| x - x_0\| \leq r\}.$$

We then call r the radius of Q. We will denote by C constants that depend only on the euclidean dimension and the Orlicz-function Φ. Constants may have varying values at different times.

We will give a mapping $f\colon Q_2 \to \mathbb{R}^n$, $Q_2 = Q(0,2)$, so that $J(x,f) < 0$ a.e. and so that the rest of the requirements hold; the desired mapping for Q_2 is then obtained by employing an auxiliary reflection in a hyperplane. The case of a general cube reduces to this by scaling.

First we introduce a sequence of compact sets in the unit cube $Q_0 = \{x \in \mathbb{R}^n : ||x|| \leq \frac{1}{2}\}$ whose intersection is a Cantor set.

The unit cube Q_0 is first divided into 2^n cubes with radius $1/4$, which are each in turn divided into a subcube with radius $(1/4)/2$ and a difference of two cubes which we refer to as an annulus. The family $\mathcal{Q}_1$ consists of these 2^n subcubes. The remainder of the construction is then self-similar. The subcube is divided into 2^n cubes which are each in turn divided into a subcube with radius $4^{-2}/2$ and an annulus. The family $\mathcal{Q}_2$ consists of these 2^{2n} subcubes (see Figure 1). Continuing this way, we get the families $\mathcal{Q}_k$, $k = 1, 2, 3, \ldots$, for which the radius of $Q \in \mathcal{Q}_k$ is $r(Q) = r_k = 2^{-2k-1}$ and the number of cubes in $\mathcal{Q}_k$ is $\#\mathcal{Q}_k = 2^{nk}$. It easily follows that the resulting Cantor set is of measure zero.



$\mathcal{Q}_1$ $\mathcal{Q}_2$

Figure 1. Families $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

Next we take a Whitney decomposition around the origin so that the resulting cubes cover the set $Q_2 \setminus \{0\}$: the first series of Whitney cubes consists of the closures of those dyadic cubes (from now on, we call the closures also dyadic) with radius $\frac{1}{2}$ whose union covers the annulus $Q_2 \setminus Q(0,1)$, and then the jth series consists of $2^{2n} - 2^n$ cubes with radius 2^{-j}, $j = 1, 2, \ldots$. We make the previous Cantor construction in each of these cubes, using the scaling factor 2^{-j}. The construction of our mapping will be the same for each of our cubes of radius 2^{-j}, modulo translations. We will describe the construction for a cube of radius 2^{-j}, centered at the origin.

The common radius of all cubes of the kth generation of the Cantor construction is $r_{jk} = 2^{-j-2k-1}$, and there are $2^{nk} = \#\mathcal{Q}_{jk}$ of them. We consider positive real numbers ϵ_{jk} such that

$$\sum_{k=1}^{\infty} \epsilon_{jk} = c_j < \infty,$$

where c_j and the ϵ_{jk}s will be determined later.

Define $f_{j,0}(x) = x$, and for every $k = 1, 2, \ldots$ set

$$\varphi_{jk}(r) = \begin{cases} 2^{-j-k-1}\left(1 + \frac{2r_{jk}-r}{r_{jk}}\epsilon_{jk}\right)\prod_{i=1}^{k-1}(1+\epsilon_{ji}), & r_{jk} \le r \le 2r_{jk} \\ 2^{-j-k-1}\frac{r}{r_{jk}}\prod_{i=1}^{k}(1+\epsilon_{ji}), & 0 \le r \le r_{jk} \end{cases}$$

and

$$f_{jk}(x) = \begin{cases} f_{j,k-1}(x), & x \notin \bigcup_{Q\in\mathcal{Q}_k} 2Q, \\ f_{j,k-1}(z(Q)) + \frac{x-z(Q)}{|x-z(Q)|}\varphi_{jk}(|x-z(Q)|), & x \in 2Q,\ Q \in \mathcal{Q}_k. \end{cases}$$

Here $z(Q)$ is the center of the cube Q.

Now, since the series $\sum_k \epsilon_{jk}$ converges, the infinite product of the terms $1+\epsilon_{jk}$ converges as well:

$$\prod_{k=j}^{\infty}(1+\epsilon_{jk}) = C'_j < \infty.$$

Thus the sequence $(f_{jk})_{k=1}^{\infty}$ converges uniformly to a limit mapping f_j. Notice that f_j fixes the boundary of our cube centered at the origin.

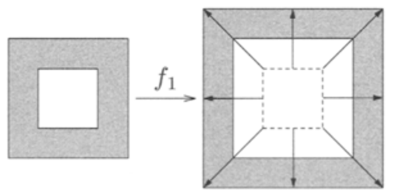


Figure 2. The mapping f_1 acting on $2Q$, $Q \in \mathcal{Q}_1$.

We do this for all j and produce the mapping f as described above. Now, f is absolutely continuous on almost all lines parallel to the coordinate axes, and $J(x,f) < 0$ for almost every $x \in Q_2$. In addition, f is continuous, since every f_j is a uniform limit of continuous mappings and the f_js keep boundaries fixed. We have to show that

(i) $\int_K |Df| < \infty$ for all compact $K \subset Q_2 \setminus \{0\}$,
(ii) $\int_{Q_2} \exp(\Phi(K)) < \infty$,
(iii) $J(\cdot, f) \in L^1_{\mathrm{loc}}(Q_2 \setminus \{0\})$, but
(iv) $J(\cdot, f) \notin L^1_{\mathrm{loc}}(Q_2)$, and
(v) f is bounded.

Fix one of the cubes Q_j in the jth series of the Whitney decomposition. Furthermore, fix one of the cubes in the kth generation of the Cantor construction inside Q_j. The mapping f is radial in the annulus $\mathrm{int}(2Q \setminus Q)$, with respect to the cubic metric;

$$f(x) = \frac{x}{||x||}\varphi_{jk}(||x||).$$

Recall (cf. [8]) that for $f(x) = \frac{x}{|x|}\varphi(|x|)$, with φ radial,

$$Df(x) = \frac{\varphi(|x|)}{|x|}\mathbf{I} + \Big(\varphi'(|x|) - \frac{\varphi(|x|)}{|x|}\Big)\frac{x \otimes x}{|x|^2},$$

where $x \otimes x$ is the $n \times n$ matrix whose i,j-entry equals $x_i x_j$, and

$$J(x,f) = \varphi'(|x|)\Big(\frac{\varphi(|x|)}{|x|}\Big)^{n-1}.$$

Thus we obtain the estimates

$$|Df(x)| \approx \max\Big\{\frac{\varphi_{jk}(||x||)}{||x||}, |\varphi'_{jk}(||x||)|\Big\} \tag{3.1}$$

and

$$J(x,f) \approx \frac{\varphi'_{jk}(||x||)\varphi_{jk}^{n-1}(||x||)}{||x||^{n-1}} \tag{3.2}$$

for almost every $x \in 2Q \setminus Q$. Here $\approx$ means that the right hand side is bounded by the left hand side from above and below, with constants not depending on the indices j and k. Furthermore

$$\begin{aligned} K(x) =& \frac{|Df(x)|^n}{|J(x,f)|} \leq C\frac{\varphi_{jk}(||x||)}{||x|||\varphi'_{jk}(||x||)|} + C\Big(\frac{||x|||\varphi'_{jk}(||x||)|}{\varphi_{jk}(||x||)}\Big)^{n-1} \\ \leq& C(\epsilon_{jk}^{-1} + \epsilon_{jk}^{n-1}). \end{aligned} \tag{3.3}$$

Let us first show (i). It suffices to show that for all fixed j_0, the modulus of the differential is integrable over the union A_{j_0} of all Whitney cubes up to j_0th series. By equation (3.1),

$$\begin{aligned} |Df(x)| &\leq C\frac{\varphi_{jk}(||x||)}{||x||} + C|\varphi'_{jk}(||x||)| \\ &\leq C2^{-j-k-1}\Big(\frac{1+\epsilon_{jk}}{r_{jk}} + \frac{\epsilon_{jk}}{r_{jk}}\Big)\prod_{i=1}^{k-1}(1+\epsilon_{ji}) \leq C2^{k+1}\prod_{i=1}^{k}(1+\epsilon_{ji}) \end{aligned}$$

for almost every $x \in 2Q \setminus Q$ for Q of radius $2^{-j-2k-1}$. Now, since the integral of the modulus of the differential is same in every cube Q_j in the jth Whitney series, we have

$$\begin{aligned} \int_{A_{j_0}} |Df(x)| =& \sum_{j=1}^{j_0}(2^{2n}-2^n)\int_{Q_j}|Df(x)| = \sum_{j=1}^{j_0}(2^{2n}-2^n)\sum_{k=1}^{\infty}2^{nk}\cdot\int_{Q_{jk}}|Df(x)| \\ \leq& C\sum_{j=1}^{j_0}\sum_{k=1}^{\infty}2^{-jn+(1-n)k}\prod_{i=1}^{k}(1+\epsilon_{ji}) \\ \leq& C\sum_{j=1}^{j_0}C(j)2^{-jn}\sum_{k=1}^{\infty}2^{(1-n)k} < \infty, \end{aligned}$$

since $n > 1$. Thus (i) holds, as long as numbers ϵ_{jk} are defined as promised before.

By equations (3.2) and (3.3) we have

$$\int_{Q_2} \exp(\Phi(K(x))) \leq C\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} 2^{-nj-nk}\exp(\Phi(C(\epsilon_{jk}^{-1}+\epsilon_{jk}^{n-1}))), \tag{3.5}$$

and

$$|J(x,f)| \leq C2^{nk}(1+\epsilon_{jk})^{n-1}\epsilon_{jk}\prod_{i=1}^{k-1}(1+\epsilon_{ji})^n \leq 2^{nk}\epsilon_{jk}\prod_{i=1}^{k}(1+\epsilon_{ji})^n,$$

$$|J(x,f)| \geq C2^{nk}\epsilon_{jk}\prod_{i=1}^{k-1}(1+\epsilon_{ji})^n.$$

Integrating, as in equation (3.4), over the set A_{j_0}, we arrive at

$$\int_{A_{j_0}} |J(x,f)| \leq C\sum_{j=1}^{j_0}\sum_{k=1}^{\infty} 2^{nk}\int_{Q_k}|J(x,f)| \leq C\sum_{j=1}^{j_0}2^{-jn}\sum_{k=1}^{\infty}\epsilon_{jk}\prod_{i=1}^{k}(1+\epsilon_{ji})^n$$
$$\leq C\sum_{j=1}^{j_0}2^{-jn}C(j)\sum_{k=1}^{\infty}\epsilon_{jk} < \infty.$$

Hence the Jacobian is locally integrable outside the origin and (iii) holds. Now,

$$\int_{Q_2} |J(x,f)| \geq C\sum_{j=1}^{\infty}2^{-jn}\sum_{k=1}^{\infty}\epsilon_{jk}\prod_{i=1}^{k-1}(1+\epsilon_{ji})^n. \tag{3.6}$$

On the other hand, for every Whitney cube Q in the j:th series the following holds:

$$|f(x)| = |\lim_{k\to\infty} f_{jk}(x)| \leq 1+\sum_{k=1}^{\infty}2^{-j-k-1}\prod_{i=1}^{k}(1+\epsilon_{ji}) \quad \text{for every } x\in Q. \tag{3.7}$$

By equation (3.3) there exists an $L>0$ such that $K(x) \leq L/\epsilon_{jk}$ for every $x \in Q_2$ as long as $\epsilon_{jk} \leq C'$ for all $j,k \in \mathbb{N}$ and some fixed $C'>0$. Now choose a sequence $(\tilde{\epsilon}_k)$ by setting

$$\tilde{\epsilon}_k = \frac{L}{\Phi^{-1}(k)}.$$

By the change of variables $s=\Phi^{-1}(t)$ in assumption (1.3), we have

$$\int_1^{\infty}\frac{dt}{\Phi^{-1}(t)} < \infty.$$

Thus the sum $\sum_{k=1}^{\infty}\tilde{\epsilon}_k$, as well as the corresponding product converge.

Next we choose for every $j \in \mathbb{N}$ an index $k(j)$ so large that

$$2^{-k}\exp(\Phi(L(2^{-j}+2^{j(n-1)}))) \leq 1.$$

Now we are ready to define the numbers ϵ_{jk}. We set

$$\epsilon_{jk} = \begin{cases} 2^j, & k = k(j), \\ 1, & k = k(j)+1, \\ \tilde{\epsilon_k}, & \text{otherwise.} \end{cases}$$

Now, as the ϵ_{jk}s are equal to $\tilde{\epsilon}_k$s except that there is a "blow-up" term for each j, the sums $\sum_{k=1}^{\infty}\epsilon_{jk}$ as well as the corresponding product converge for each j.

Combining the previous choices for ϵ_{jk}s with estimate (3.5), we have

$$\begin{aligned}\int_{Q_2}\exp(\Phi(K(x))) &\leq C\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}2^{-nj-nk}\exp(\Phi(C'(\epsilon_{jk}^{-1}+\epsilon_{jk}^{n-1})))\\ &\leq C\sum_{j=1}^{\infty}2^{-nj}\sum_{k=1}^{\infty}2^{-nk}\exp\Big(\Phi\Big(\frac{L}{\tilde{\epsilon}_k}\Big)\Big)+\sum_{j=1}^{\infty}2^{-nj}C(1+\exp(\Phi(C)))\\ &\leq C+C\sum_{j=1}^{\infty}2^{-nj}\sum_{k=1}^{\infty}2^{-nk}\exp(k) = C'+\sum_{j=1}^{\infty}2^{-nj}\sum_{k=1}^{\infty}(2^{-n}e)^k<\infty.\end{aligned}$$

Thus (ii) holds. Furthermore, estimate (3.6) yields

$$\int_{Q_2}|J(x,f)| \geq C\sum_{j=1}^{\infty}2^{-jn}\sum_{k=1}^{\infty}\epsilon_{jk}\prod_{i=1}^{k-1}(1+\epsilon_{ji})^n \geq C\sum_{j=1}^{\infty}2^{-jn}(2^{jn}) = \infty,$$

which proves (iv). Finally, using (3.7), we obtain

$$|f(x)| = |\lim_{k\to\infty}f_{jk}(x)| \leq 1+\sum_{k=1}^{\infty}2^{-j-k-1}\prod_{i=1}^{k}(1+\epsilon_{ji}) \leq 1+C'\sum_{k=1}^{\infty}2^{-k}$$

for every $x \in Q$, Q being any of the Whitney cubes with radius 2^{-j}. Here the constant C' does not depend on j, and hence (v) is verified. This finishes the proof. ∎

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